DETERMINING EQUATION FOR RUBBERLIKE THERMOVISCOELASTIC MATERIALS WITH FINITE STRAINS

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Models of rubberlike bodies are constructed through splitting of the tensor of total final strains of the material into tensors of thermal strains and elastic strains, respectively, according to the thermodynamics of irreversible processes.

Many rubberlike materials admitting larger reversible strains are quite accurately described by the viscoelastic model (Kelvin body) with inclusion of thermal strains [1-3]. Here the total displacements u of particles of a material are represented as sums of a thermal component u_t and an elastic component u_e, just as in the case of an elastoplastic medium [4]. Such an approach makes it possible to demonstrate that thermal and elastic strains are incompatible apart, that the total strain of a medium with a Cauchy measure is equal to the product of the tensor of the thermal displacement by the tensor of the elastic displacement components, and that the relative density of a material can be represented as the product of the thermal relative density by the elastic relative density.

Three rheological models (Fig. 1a, b, c) have been proposed for describing the properties of rubberlike materials. The rectangles with the letters E, V, T represent respectively the elasticity element, the viscosity element, and the thermal element. Element T accounts only for thermal strains and does not affect the stress tensor in a rheological model. It will be demonstrated here that, if in the determining equations for model 1b one approaches the limit where the elastic strains in element E_2 approach zero and the elasticity coefficients then change so as to leave stresses finite, model 1a will at the limit approach model 1b. It follows from here that model 1a will not be thermodynamically inconsistent, if regarded as the limiting case of model 1b in the given sense. Let us construct the determining equations for the rheological model 1a.

1. Let the location of a particle of a rubberlike material in its initial undeformed state 0 at the initial temperature T_0 be defined by vector $\mathbf{r}_0$ and its location in the deformed state 2 at the temperature T be defined by the vector $\mathbf{r}_2$. Fictitiously we divide the entire body into small particles and define the intermediate state of stress 1 of each particle at the initial temperature T_0 by vector $\mathbf{r}_1$. The transition from state 0 to state 1 can be characterized by an elastic displacement u_e and the transition from state 1 to the final state 2 can be characterized by a thermal displacement u_t. The total displacement u of particles of the material from state 0 to 2 will be related to displacements u_t and u_e through the equality

$$\mathbf{u} = \mathbf{u}_t + \mathbf{u}_e. \tag{1}$$

The thermal strains of the material are assumed to be isotropic. The tensors C of final strains, C_t of thermal strains, and C_e of elastic strains of the material will be expressed through corresponding Cauchy tensors in Euler variables

$$\mathbf{Э}_t = \lim \partial \mathbf{r}_1/\partial \mathbf{r}_2, \quad \mathbf{Э}_e = \lim \partial \mathbf{r}_0/\partial \mathbf{r}_1, \quad \mathbf{Э} = \partial \mathbf{r}_0/\partial \mathbf{r}_2 = \mathbf{Э}_e \cdot \mathbf{Э}_t, \tag{2}$$
$$\mathbf{C} = \mathbf{Э}^* \cdot \mathbf{Э}, \quad \mathbf{C}_t = \mathbf{Э}_t^* \cdot \mathbf{Э}_t, \quad \mathbf{C}_e = \mathbf{Э}_e^* \cdot \mathbf{Э}_e.$$

The symbol "lim" denotes transition to the limit in the fictitious infinitesimal comminution of particles of the material for the purpose of reaching state 1 from state 2 through a formal change of the temperature T to its initial level T_0. Each field of displacements u_t and u_e is incompatible and forms a nonequivalent space. Therefore tensors C_t and C_e separately cannot satisfy the equations of compatibility. The tensors of the rates of total, elastic, and thermal strains will be introduced here in the form

$$\boldsymbol{\varepsilon} = -\operatorname{sym}(\mathbf{Э}^{-1} \cdot \mathbf{Э}^{\cdot}) = \operatorname{sym}(\operatorname{grad} \mathbf{v}) = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_t, \tag{3}$$
$$\boldsymbol{\varepsilon}_e = -\operatorname{sym}(\mathbf{Э}_e^{-1} \cdot \mathbf{Э}^{\cdot}), \quad \boldsymbol{\varepsilon}_t = -\mathbf{Э}_t^{-1} \cdot \mathbf{Э}_t^{\cdot} \mathbf{I},$$

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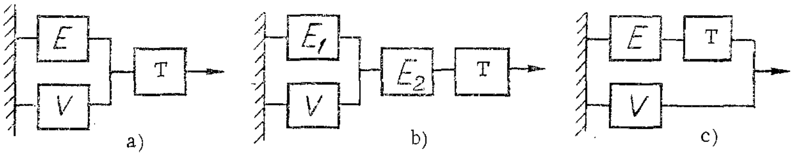

Fig. 1. Simplest rheological models of rubberlike materials with thermal strains resulting (a, b) and not resulting (c) in internal viscous stresses.

where v is the velocity of particles of the material.

Without detracting from the generality, one can regard the tensor $Э_t$ as a spherical one

$$Э_t = Э_t \cdot \mathbf{I}, \tag{4}$$

where $Э_t$ is the scalar function of thermal expansion. Applying the theorem of polar expansion to tensors Э and $Э_e$, then using expression (4) and the equality $Э = Э_t \cdot Э_e$, we first establish the relation between the tensor of final strains of the material and the tensors of its elastic strains and of its thermal strains

$$\mathbf{C} = \mathbf{C}_t \cdot \mathbf{C}_e. \tag{5}$$

Let R_e, R_t, and R denote the relative changes in the density of the material upon transitions 0-1, 1-2, and 0-2, respectively. These relative changes in density are determined by the equalities

$$R_e = \det Э_e, \quad R_t = Э_t^3, \quad R = \det Э = R_t R_e. \tag{6}$$

From the equation of mass conservation and from the independence of R_t and R_e we find, with the aid of relations (6), that R_t and R_e satisfy the individual equations of mass conservation

$$R_t + R_t (\boldsymbol{\varepsilon}_t : \mathbf{I}) = 0, \quad R_e + R_e (\boldsymbol{\varepsilon}_e : \mathbf{I}) = 0. \tag{7}$$

In the special case of incompressible elastic strains, relations (6), (7), and (3) yield

$$R_e = 1, \ \boldsymbol{\varepsilon}_e : \mathbf{I} = 0, \ \operatorname{div} \mathrm{v} + \frac{3}{2} \frac{d \ln \mathbf{C}_t}{dt} = 0, \ C_1 C_2 C_3 = C_t^3, \tag{8}$$

where C_i are the principal values of the C-tensor. Equalities (8) must be used for constructing a model of a rubberlike body with incompressible elastic strains.

We will assume that the internal energy U, the entropy S, and the free energy A = U − TS depend on T and on C_e. Then the Gibbs thermodynamic equation [5, 6] can be written in the form

$$dA + S dT = M_0 dT + \mathbf{M}_e : d\mathbf{C}_e. \tag{9}$$

From this, by virtue of the differentials dT and dC_e being independent, we obtain

$$S = M_0 - \partial A / \partial T, \quad \mathbf{M}_e = \partial A / \partial \mathbf{C}_e. \tag{10}$$

In order to close the system of determining equations, it is necessary to apply the first law of thermodynamics

$$\rho \dot{U} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \operatorname{div} \mathrm{q}, \tag{11}$$

where σ is the tensor of true stresses and q is the vector of thermal flux, and to use the Clausius−Duhem inequality

$$T^{-1} (\mathrm{q} \operatorname{grad} T) + \rho M_0 \dot{T} + \rho \mathbf{M}_e : \dot{\mathbf{C}}_e - \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \leqslant 0. \tag{12}$$

Note must be taken of the peculiarity of thermal strains that, unlike elastic, viscous, and plastic strains (or strain rates) in the determining equations, they are related not to stresses but to the temperature. The simplest temperature dependence is a linear one

$$\ln C_t = -2\alpha (T - T_0), \tag{13}$$

where α is the coefficient of thermal expansion. In generalizing the relation (13), one can regard α as dependent on T and C_e or in a still more general case

$$C_{\mathrm{t}} = C_{\mathrm{t}}(T,\ \mathbf{C}_e). \tag{14}$$

No experimental study has yet revealed a dependence of thermal strains on elastic strains and, therefore, such a dependence must still be regarded as a theoretical possibility only.

In the rheological model 1a the tensor of true stresses σ is equal to the sum of symmetric tensors of elastic stresses s_e and viscous stresses s_v

$$\sigma = \mathbf{s}_e + \mathbf{s}_v,\quad \mathbf{s}_e = \mathbf{s}_e(T,\ \mathbf{C}_e),\quad \mathbf{s}_v = \mathbf{s}_v(T,\ \mathbf{C}_v,\ \boldsymbol{\varepsilon}_v). \tag{15}$$

We add the auxiliary equality

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \mathbf{s}_e : \boldsymbol{\varepsilon}_e + \mathbf{s}_v : \boldsymbol{\varepsilon}_e - 1/2\,[(\mathbf{s}_e + \mathbf{s}_v)\, d \ln C_{\mathrm{t}}/dt] : \mathbf{I}. \tag{16}$$

The symmetric tensors M_e and $\partial C_t/\partial C_e$ are tensor functions of T and C_e and, therefore, commutative with C_e. Using this property and equality (16), we transform the inequality (12) to

$$\begin{gathered} T^{-1}(\mathbf{q}\cdot \operatorname{grad} T) + \left(\rho M_0 + \frac{\boldsymbol{\sigma} : \mathbf{I}}{2}\,\frac{\partial \ln C_{\mathrm{t}}}{\partial T}\right)\dot{T} - \left\{\left[2\rho \mathbf{M}_e \cdot \mathbf{C}_e\right.\right. \\ \left.\left. + (\mathbf{s}_e : \mathbf{I})\,\frac{\partial \ln C_{\mathrm{t}}}{\partial \mathbf{C}_e}\cdot \mathbf{C}_e + \mathbf{s}_e\right] : \boldsymbol{\varepsilon}_e\right\} - \left[\mathbf{s}_v + (\mathbf{s}_v : \mathbf{I})\,\frac{\partial \ln C_{\mathrm{t}}}{\partial \mathbf{C}_e}\cdot \mathbf{C}_e\right] : \boldsymbol{\varepsilon}_v \leqslant 0. \end{gathered} \tag{17}$$

An analysis of the thermodynamic inequality (17) yields

$$\begin{gathered} \mathbf{q} = -\Lambda \operatorname{grad} T,\ \Lambda \geqslant 0,\ M_0 = -\frac{\boldsymbol{\sigma} : \mathbf{I}}{2\rho}\,\frac{\partial \ln C_{\mathrm{t}}}{\partial T}, \\ \mathbf{s}_e = -2\rho\,\frac{\partial A}{\partial \mathbf{C}_e}\cdot \mathbf{C}_e - \frac{\partial \ln C_{\mathrm{t}}}{\partial \mathbf{C}_e}\cdot \mathbf{C}_e\,(\mathbf{s}_e : \mathbf{I}), \\ \left[\mathbf{s}_v - (\mathbf{s}_v : \mathbf{I})\,\frac{\partial \ln C_{\mathrm{t}}}{\partial \mathbf{C}_e}\cdot \mathbf{C}_e\right] : \boldsymbol{\varepsilon}_v \geqslant 0. \end{gathered} \tag{18}$$

Resolving the third of equalities (18) with respect to s_e, we obtain

$$\mathbf{s}_e = 2\rho\,\frac{\partial \ln C_{\mathrm{t}}}{\partial \mathbf{C}_e}\cdot \mathbf{C}_e\left(\frac{\partial A}{\partial \mathbf{C}_e} : \mathbf{C}_e\right)\left[1 + \left(\frac{\partial \ln C_{\mathrm{t}}}{\partial \mathbf{C}_e} : \mathbf{C}_e\right)\right]^{-1} - 2\rho\,\frac{\partial A}{\partial \mathbf{C}_e}\cdot \mathbf{C}_e. \tag{19}$$

After the expression (18) for M_0 has been inserted into the equality (10) for S, it becomes evident that the state function S depends not only on T and C_e but also on the invariant σ:I and consequently on ε_v. It follows immediately that the internal energy U can also depend not only on T and C_e but also on ε_v, which contradicts the thermodynamic definitions of internal energy and of entropy. This inconsistency of relations (18) can be removed in various ways. One can assume

$$\mathbf{s}_v : \mathbf{I} = 0, \tag{20}$$

with no bulk viscosity, e.g., which is sometimes done in the case of a viscous gas. A drawback of condition (20) is that in the case of anisotropic thermal strains an assumption analogous to (20) would lead to the condition $s_v = 0$, i.e., to the absence of the viscous element in the model 1a.

We introduce the notation

$$\mathbf{N}_a = \mathbf{s}_v + (\mathbf{s}_v : \mathbf{I})\,\mathbf{C}_e\cdot \partial \ln C_{\mathrm{t}}/\partial \mathbf{C}_e. \tag{21}$$

Resolving expression (21) with respect to s_v, we obtain

$$\mathbf{s}_v = \mathbf{N}_a - (\mathbf{N}_a : \mathbf{I})\,[1 + \mathbf{C}_e : \partial \ln C_{\mathrm{t}}/\partial \mathbf{C}_e]^{-1}\,\mathbf{C}_e\cdot \partial \ln C_{\mathrm{t}}/\partial \mathbf{C}_e. \tag{22}$$

Expression (21) transforms the last of inequalities (18) to

$$\mathbf{N}_a(T,\ \mathbf{C}_e,\ \boldsymbol{\varepsilon}_v) : \boldsymbol{\varepsilon}_v \geqslant 0. \tag{23}$$

The simplest dependence of the tensor function N_a on ε_v which satisfies inequality (23) is

$$\mathbf{N}_a = \xi_a(T,\ \mathbf{C}_e)\,(\boldsymbol{\varepsilon}_v : \mathbf{I})\,\mathbf{I} + 2\eta_a(T,\ \mathbf{C}_e)\,\boldsymbol{\varepsilon}_v,\ \xi_a \geqslant 0,\ \eta_a \geqslant 0, \tag{24}$$

where ξ_a and η_a are viscosity coefficients. When condition (20) is satisfied, then the equality $s_v = N_a$, $\xi_a = 0$ will hold true.

2. We will now demonstrate that the determining equations for the model 1b are not thermodynamically inconsistent and that there is an ultimate expression for them which corresponds to an elasticity element E_2 where strains have vanished. Then, formally, model 1b becomes model 1a. The deformation of the material according to the model 1b is determined by the tensors $\mathbf{Э}_1$, $\mathbf{Э}_2$ and $\mathbf{Э}_t$, similar to the tensors (2). Without detracting from the generality, one can regard the tensor $\mathbf{Э}_2$, as asymmetric one and the thermal strains as isotropic ones. The tensor $\mathbf{Э}_t$ will be a spherical one

$$\mathbf{Э}_t = C_t^{1/2}\mathbf{I}, \quad \mathbf{Э}_2 = \overset{*}{\mathbf{Э}}_2 = \mathbf{C}_2^{1/2}.$$

The tensors of final strains C_1, C_2, C_t and the tensors of strain rates ε_1, ε_2, ε_t in the rheological elements E_1, E_2, and T are

$$\begin{gathered} \mathbf{C}_1 = \overset{*}{\mathbf{Э}}_1 \cdot \mathbf{Э}_1, \quad \mathbf{C}_2 = \overset{*}{\mathbf{Э}}_2 \cdot \mathbf{Э}_2, \quad \mathbf{C}_t = \mathcal{Э}_t^2\,\mathbf{I}, \\ \boldsymbol{\varepsilon}_2 = -\operatorname{sym}(\mathbf{Э}_2^{-1} \cdot \dot{\mathbf{Э}}_2 + \mathbf{Э}_2^{-1} \cdot \mathbf{Э}_1^{-1} \cdot \dot{\mathbf{Э}}_1 \cdot \mathbf{Э}_2 - \mathbf{Э}_1^{-1} \cdot \dot{\mathbf{Э}}_1), \\ \boldsymbol{\varepsilon}_t = -\mathcal{Э}_t^{-1}\dot{\mathcal{Э}}_t\,\mathbf{I}. \end{gathered} \tag{25}$$

The expressions for the tensors ε_1, ε_2 have the form of a recurrence relation and satisfy the equality

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_t, \tag{26}$$

where ε is the tensor of total strain rate in the material. In the rheological model 1b hold true the relations

$$\boldsymbol{\sigma} = \mathbf{s}_1 + \mathbf{s}_v = \mathbf{s}_2, \quad \boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_v, \quad \mathbf{C}_1 = \mathbf{C}_v. \tag{27}$$

Just as in the model 1a, we assume here that the state functions U, S, and A depend on T, C_1, and C_2. We write the second law of thermodynamics as

$$\begin{gathered} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \rho M_0 \dot{T} - \rho \mathbf{M}_2 : \dot{\mathbf{C}}_2 - \rho \mathbf{M}_1 : \dot{\mathbf{C}}_1 - T^{-1}(\mathbf{q}\operatorname{grad} T) \geqslant 0, \\ S = M_0 - \partial A/\partial T, \quad \mathbf{M}_i = \partial A/\partial \mathbf{C}_i \ (i = 1,\ 2). \end{gathered} \tag{28}$$

The first term $\sigma : \varepsilon$ in expression (28) we then transform to

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \mathbf{s}_1 : \boldsymbol{\varepsilon}_1 + \mathbf{s}_v : \boldsymbol{\varepsilon}_1 + \mathbf{s}_2 : \boldsymbol{\varepsilon}_2 + \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t. \tag{29}$$

By virtue of relations (25) and (29), the inequality (28) can be split into two expressions

$$\begin{gathered} \mathbf{q} = -\Lambda \operatorname{grad} T, \quad \Lambda \geqslant 0, \\ \mathbf{C}_2^{1/2} \cdot \boldsymbol{\sigma} \cdot \mathbf{C}_2^{-1/2} + 2\rho \frac{\partial A}{\partial \mathbf{C}_1} \cdot \mathbf{C}_1 + (\boldsymbol{\sigma} : \mathbf{I}) \frac{\partial \ln C_t}{\partial \mathbf{C}_1} : \boldsymbol{\varepsilon}_1 - \\ - \left[\mathbf{s}_2 + 2\rho \frac{\partial A}{\partial \mathbf{C}_2} \cdot \mathbf{C}_2 + (\mathbf{s}_2 : \mathbf{I}) \frac{\partial \ln C_t}{\partial \mathbf{C}_2} \cdot \mathbf{C}_2 \right] : \mathbf{Э}_2^{-1} \cdot \dot{\mathbf{Э}}_2 - \left[\rho M_0 + \frac{(\boldsymbol{\sigma} : \mathbf{I})}{2} \frac{\partial \ln C_t}{\partial T} \right] \dot{T} \geqslant 0. \end{gathered} \tag{30}$$

The coefficient of $\dot{T}$ inside brackets in inequality (30) does not depend on $\dot{T}$. Neither do all other terms in this expression and, therefore, this coefficient must become equal to zero

$$\rho M_0 + \frac{(\mathbf{s}_2 : \mathbf{I})}{2} \frac{\partial \ln C_t}{\partial T} = 0, \ \text{i.e.,} \ S = -\frac{\partial A}{\partial T} - \frac{(\mathbf{s}_2 : \mathbf{I})}{2\rho} \frac{\partial \ln C_t}{\partial T}. \tag{31}$$

The coefficient of $\mathbf{Э}_2^{-1} \cdot \dot{\mathbf{Э}}_2$ inside brackets in inequality (30) does not depend on the tensors of strain rates ε_1 and ε_2, since it is determined only by the properties of the elastic element E_2. Assuming that also all remaining terms in expression (30) do not depend on $\mathbf{Э}_2^{-1} \cdot \dot{\mathbf{Э}}_2$, we conclude that also this coefficient must become equal to zero

$$\mathbf{s}_2 + (\mathbf{s}_2 : \mathbf{I}) \frac{\partial \ln C_t}{\partial \mathbf{C}_2} \cdot \mathbf{C}_2 + 2\rho \frac{\partial A}{\partial \mathbf{C}_2} \cdot \mathbf{C}_2 = 0. \tag{32}$$

Expression (32) yields

$$\boldsymbol{\sigma} = \mathbf{s}_2 = 2\rho \left(\frac{\partial A}{\partial \mathbf{C}_2} : \mathbf{C}_2 \right) \left[1 + \left(\frac{\partial \ln C_t}{\partial \mathbf{C}_2} : \mathbf{C}_2 \right) \right]^{-1} \frac{\partial \ln C_t}{\partial \mathbf{C}_2} \cdot \mathbf{C}_2 - 2\rho \frac{\partial A}{\partial \mathbf{C}_2} \cdot \mathbf{C}_2. \tag{33}$$

The thus simplified inequality (30), on the basis of expressions (31) and (32), can be written in the form

$$\left[\mathbf{C}_2^{1/2}\cdot\mathbf{s}_1\cdot\mathbf{C}_2^{-1/2}+2\rho\frac{\partial A}{\partial\mathbf{C}_1}\cdot\mathbf{C}_1+(\mathbf{s}_1:\mathbf{I})\;\frac{\partial\ln C_{\mathrm{t}}}{\partial\mathbf{C}_1}\cdot\mathbf{C}_1\right]:\boldsymbol{\varepsilon}_1+\left[\mathbf{C}_2^{1/2}\cdot\mathbf{s}_v\cdot\mathbf{C}_2^{-1/2}+(\mathbf{s}_v:\mathbf{I})\;\frac{\partial\ln C_{\mathrm{t}}}{\partial\mathbf{C}_1}\cdot\mathbf{C}_1\right]:\boldsymbol{\varepsilon}_v\geqslant 0. \tag{34}$$

The value of the coefficient of ε_1 in expression (34) is determined by the elastic properties of the rheological element E_1, these properties depending on ε_1. Therefore, equating this coefficient to zero will yield an equation describing the properties of the rheological elasticity element E_1, namely

$$\mathbf{s}_1=2\rho\mathbf{C}_2^{-1/2}\cdot\left[\left(\frac{\partial A}{\partial\mathbf{C}_1}:\mathbf{C}_1\right)\left(1+\frac{\partial\ln C_{\mathrm{t}}}{\partial\mathbf{C}_1}:\mathbf{C}_1\right)^{-1}\frac{\partial\ln C_{\mathrm{t}}}{\partial\mathbf{C}_1}\cdot\mathbf{C}_1-\frac{\partial A}{\partial\mathbf{C}_1}\cdot\mathbf{C}_1\right]\cdot\mathbf{C}_2^{1/2}. \tag{35}$$

With the aid of this equation, we now rewrite inequality (34) as

$$\mathbf{N}_b(T,\ \mathbf{C}_1,\ \mathbf{C}_2,\ \boldsymbol{\varepsilon}_v):\boldsymbol{\varepsilon}_v\geqslant 0, \tag{36}$$

where

$$\mathbf{N}_b=\mathbf{C}_2^{1/2}\cdot\mathbf{s}_v\cdot\mathbf{C}_2^{-1/2}+(\mathbf{s}_v:\mathbf{I})\,\mathbf{C}_1\cdot\partial\ln C_{\mathrm{t}}/\partial\mathbf{C}_1.$$

The tensor of stresses s_V in the viscosity element V can be expressed through the tensor of stresses $\mathbf{N}_b$ in the form

$$\mathbf{s}_v=\mathbf{C}_2^{-1/2}\cdot\left[\mathbf{N}_b-(\mathbf{N}_b:\mathbf{I})\left(1+\frac{\partial\ln C_{\mathrm{t}}}{\partial\mathbf{C}_1}:\mathbf{C}_1\right)^{-1}\frac{\partial\ln C_{\mathrm{t}}}{\partial\mathbf{C}_1}\cdot\mathbf{C}_1\right]\cdot\mathbf{C}_2^{1/2}. \tag{37}$$

The simplest dependence of the N_b-tensor on ε_V which satisfies inequality (36) is

$$\mathbf{N}_b=\xi_b(T,\ \mathbf{C}_1,\ \mathbf{C}_2)(\boldsymbol{\varepsilon}_v:\mathbf{I})\,\mathbf{I}+2\eta_b(T,\ \mathbf{C}_1,\ \mathbf{C}_2)\,\boldsymbol{\varepsilon}_v,\ \ \xi_b\geqslant 0,\ \ \eta_b\geqslant 0, \tag{38}$$

where ξ_b and η_b are viscosity coefficients.

The system (31), (33), (35), (37), (38) constitutes the determining equations for the model 1b. In contrast to the preceding case, here the entropy (31) is, according to relation (33), a state function so that the model 1b, unlike the model 1a, is not thermodynamically inconsistent.

The model 1a can be obtained from the model 1b through passage to the limit where strains in the elasticity element E_2 vanish. In the passage to the limit it is necessary, at the same time, that the elastic properties of element E_2 which appear in the expression for the free energy A assume those limiting values which will ensure the equality $\sigma = s_2$ (33), where the tensor of true stresses σ has been assumed to have some fixed value

$$\mathbf{C}_2\to\mathbf{I}\ \text{when}\ \mathbf{s}_2=\boldsymbol{\sigma}=\text{const}. \tag{39}$$

The entropy in the expression for it always remains a state function during passage to the limit, but not any more in the ultimate expression (10) for it. All other determining equations for the model 1b become at the limit the corresponding equations for the model 1a.

3. Thermal strains of a classical viscous fluid are accounted for only in the equation which relates the hydrostatic pressure to the density and the temperature. Its viscous properties, however, such a fluid exhibits immediately with the appearance of a strain rate and regardless of what has produced it, whether mechanical action or heating. Such a manifestation of the viscous properties can be incorporated in a rheological model describing the properties of rubberlike materials. A model like this is shown in Fig. 1c.

The deformation of the material is in this model determined by the tensors Э and Э_{t} (2), the Э_{t}-tensor being a spherical one. The state functions U, S, and A depend on T and C_1 so that the second law of thermodynamics appears in the form

$$\boldsymbol{\sigma}:\boldsymbol{\varepsilon}-\rho M_0\dot{T}-\rho\mathbf{M}_1:\dot{\mathbf{C}}_1-T^{-1}(\mathbf{q}\cdot\operatorname{grad}T)\geqslant 0. \tag{40}$$

The $\sigma:\varepsilon$ term in expression (40) for this model transforms to

$$\boldsymbol{\sigma}:\boldsymbol{\varepsilon}=\mathbf{s}_1:\boldsymbol{\varepsilon}_1+\mathbf{s}_1:\boldsymbol{\varepsilon}_{\mathrm{t}}+\mathbf{s}_v:\boldsymbol{\varepsilon},\ \text{where}\ \mathbf{s}_1+\mathbf{s}_v=\boldsymbol{\sigma} \tag{41}$$

Inserting expressions (2), (3), and (41) into expression (40) yields the inequality

$$\mathbf{s}_v : \boldsymbol{\varepsilon} + \mathbf{s}_1 : \boldsymbol{\varepsilon}_1 + (\mathbf{s}_1 : \mathbf{I}) \frac{\partial \ln C_t}{\partial \mathbf{C}_1} \cdot \mathbf{C}_1 : \boldsymbol{\varepsilon}_1 + 2\rho \mathbf{M}_1 \cdot \mathbf{C}_1 : \boldsymbol{\varepsilon}_1 - \rho M_0 \dot{T} - \frac{(\mathbf{s}_1 : \mathbf{I})}{2} \frac{\partial \ln C_t}{\partial T} \cdot \dot{T} - T^{-1} (\mathbf{q} \cdot \operatorname{grad} T) \geqslant 0. \quad (42)$$

This inequality, after appropriate simplifications analogous to those made in inequality (28), splits into the relations

$$\mathbf{q} = -\Lambda \operatorname{grad} T, \ \Lambda \geqslant 0, \ S = -\frac{\partial A}{\partial T} - \frac{(\mathbf{s}_1 : \mathbf{I})}{2\rho} \frac{\partial \ln C_t}{\partial T}, \quad (43)$$

$$\mathbf{s}_1 = 2\rho \left(\frac{\partial A}{\partial \mathbf{C}_1} : \mathbf{C}_1 \right) \left(1 + \frac{\partial \ln C_t}{\partial \mathbf{C}_1} : \mathbf{C}_1 \right)^{-1} \frac{\partial \ln C_t}{\partial \mathbf{C}_1} \cdot \mathbf{C}_1 - 2\rho \frac{\partial A}{\partial \mathbf{C}_1} \cdot \mathbf{C}_1, \quad \mathbf{s}_v : \varepsilon \geqslant 0. \quad (44)$$

The simplest linear dependence of the s_V-tensor on ε which satisfies inequality (44) is

$$\mathbf{s}_v = \xi (T, \mathbf{C}_1) (\boldsymbol{\varepsilon} : \mathbf{I}) \mathbf{I} + 2\eta (T, \mathbf{C}_1) \boldsymbol{\varepsilon}, \ \xi \geqslant 0, \ \eta \geqslant 0. \quad (45)$$

Relations (43)-(45) are the sought determining equations for this model. They can be greatly simplified by a change from the independent thermodynamic variables (T, C_1) to the variables (T, C). After this change, Eqs. (43)-(45) become

$$\begin{gathered} \mathbf{q} = -\Lambda \operatorname{grad} T, \ \Lambda \geqslant 0, \ A = A(T, \mathbf{C}), \ S = -\partial A / \partial T, \\ \mathbf{s}_1 = -2\rho \mathbf{C} \cdot \partial A / \partial \mathbf{C}, \quad \mathbf{s}_v = \mathbf{s}_v (T, \mathbf{C}, \boldsymbol{\varepsilon}), \quad \mathbf{s}_v : \boldsymbol{\varepsilon} \geqslant 0. \end{gathered} \quad (46)$$

There arises the question: which of the models considered here is preferable? The models 1b and 1c differ in that the total strain due to instantaneous heating (thermal shock) of a free beam without load is in the first case equal to the instantaneously appearing thermal strain, which then does not change with time, while in the second case it will relax, i.e., change with time from its initial magnitude to that of the thermal strain. On the basis of this difference, one can in an appropriately set up experiment establish which of these two models is suitable for describing a specific rubberlike material.

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